

# STUDY OF A MODEL OF QUANTUM ELECTRODYNAMICS <sup>1</sup>

O.W. Greenberg

*Center for Theoretical Physics*

*Department of Physics*

*University of Maryland*

*College Park, MD 20742-4111*

University of Maryland Preprint PP-00-020[5mm]

## Abstract

This paper studies the model of the quantum electrodynamics (QED) of a single nonrelativistic electron due to W. Pauli and M. Fierz and studied further by P. Blanchard. This model exhibits infrared divergence in a very simple context. The infrared divergence is associated with the inequivalence of the Hilbert spaces associated with the free Hamiltonian and with the complete Hamiltonian. Infrared divergences that are visible in the perturbative description disappear in the space of the clothed electrons. In this model when the Hamiltonian is expressed in terms of the “physical” fields that create the electron together with its cloud of soft photons the variational principle suggested earlier can be applied. At finite time the Heisenberg field of the model acts in the space of the perturbative electron together with a finite number of perturbative photons, while the “physical” field can be chosen to act in the space of the exact (“physical”) electron eigenstates together with a finite number of physical photons. The space of the physical (or clothed) electron states can be chosen to be a Fock space.

## 1 Introduction

Kurt Haller has pursued fundamental issues in gauge theories. He has especially emphasized the importance of gauge invariant formulations of quantum electrodynamics and of quantum chromodynamics and of the implementation of Gauss’ law

---

<sup>1</sup>email address, owgreen@physics.umd.edu.

in these theories. Kurt uses operator methods rather than the more-popular path integral methods; this provides a good alternative to the usual point of view. I am happy to dedicate this paper to Kurt Haller.

Although there is a long history of the study of charged particles in gauge theories [1, 2, 3], there is still concern that the issue is not understood fully. For example, in a recent paper [3] the authors state “the battle is not yet won and one could, in a provocative way, summarize the situation by saying that the question ‘what is an electron in QED’ is still open.”

The purpose of this pedagogical paper is to reconsider a very simple model of QED that illustrates one facet of the infrared issues connected with charged particles. At the end I will discuss what may carry over to more realistic models of QED as well as to QED itself. The model is due to W. Pauli and M. Fierz [4]. They studied a simple, exactly soluble model of a single nonrelativistic electron in QED. P. Blanchard [5] studied this model in the interaction picture and showed that the transition operator is well-defined for finite times, but exhibits an infrared divergence for  $t \rightarrow \pm\infty$ . This model exhibits in an extremely simple context the divergence associated with infinite numbers of soft photons associated with a charged particle. The utility of this model is that explicit formulas valid to all orders in  $e$  can be found for the transformation between the perturbative states and fields and the physical ones.

I reconsider this model using the exact eigenstates of the complete Hamiltonian. I find that the Heisenberg field of the model at finite times creates states that, because of the infrared divergence, are orthogonal to the exact eigenstates. When the Hamiltonian is expressed in terms of a field that creates the exact single-particle eigenstates associated with an electron in this model, the infrared divergence disappears. I suggest that at least part of the solution to the infrared problem is to replace the original Heisenberg field by one that creates the electron with the divergent part of its soft photon cloud.

There are three effects associated with the massless quanta in gauge theories: (1) the infinite number of low-momentum quanta associated with a charged particle, (2) the collinearity of low-momentum quanta, and (3) the long-range interaction due to the exchange of massless quanta between charged particles. Here I study only

the first of these effects.

F. Bloch and A. Nordsieck [6] were the first to show how to remove infrared divergences by summing the cross sections for scattering into final states that have a charged particle together with any number of soft photons. D.R. Yennie, S.C. Frautschi and H. Suura [7] gave a general discussion of the removal of the divergences in the cross sections. P.P. Kulish and L.D. Faddeev [8] showed how to remove the divergences in the S-matrix. I want to show how to remove the divergences in the operator solution of the Pauli-Fierz model using modified asymptotic fields. In the charged single-particle model discussed here, the physical field and the modified asymptotic field are the same.

First I find the exact single-particle eigenstates of the model, which have coherent states of soft perturbative photons. Secondly, I solve the operator equations of motion of the model and find that the Heisenberg field acts in a space that is orthogonal to the space of the exact eigenstates. Thirdly, I find fields that diagonalize the Hamiltonian. Finally I discuss which of the properties of this model can be expected to be relevant to more realistic models of QED as well as to QED itself.

The exact eigenstates in this model should have the usual orthonormality properties and thus the operators that create and annihilate the exact eigenstates should have the usual free field commutation relations. This is guaranteed since the transformation between these two sets of fields is formally unitary. Thus, following Blanchard [5] and Contopanagos and Einhorn [1] I assume that the original field operators act in a Hilbert space associated with coherent states that is unitarily inequivalent to the usual Fock space, while the physical fields that create the particles together with their soft photon clouds are free fields acting in the usual Fock space. From the point of view of the usual canonical formalism this is a controversial choice. The implication of this choice is that in gauge theories and perhaps in other theories with massless particles or fields the states of relevance to observation are very distant from the states created by smeared polynomials in the fields in the original Hamiltonian or Lagrangian. We already know that in field theory the interacting fields create much more than just the particle with which they are associated. Further in theories with infinite field strength renormalization there is an infinite multiplicative constant relating the unrenormalized fields, which naively obey canonical

commutation relations, and the renormalized fields whose equal-time commutators are ill-defined. The coherent state operator for soft photons that relates the original fields of the Hamiltonian to the fields that make the single-particle eigenstates in the Pauli-Fierz model can be viewed as an operator-valued analog of the field strength renormalization.

Another purpose of this paper is to find the proper extension of the variational principle proposed earlier [9] to gauge theories. I will argue that in the Pauli-Fierz model, one should first transform to the fields that incorporate the divergent part of the soft photon cloud associated with the electron in the exact single-particle eigenstates and then apply the non-gauge theory form of the variational principle.

## 2 Exact eigenstates in the Pauli-Fierz model

The Pauli-Fierz model is nonrelativistic quantum electrodynamics in the transverse gauge with the following approximations: (1) momentum conservation between the photon and the electron is neglected<sup>2</sup>, (2) the term quadratic in the vector potential is dropped, and (3) the Coulomb interaction between electrons is dropped. So the model is a single electron interacting with massless transverse photons. The Hamiltonian is

$$\begin{aligned}
H = & \psi^\dagger(x) \left( -\frac{1}{2m} \nabla^2 + V(x) \right) \psi(x) + \sum_{s=1}^2 \int d^3k k a_s^\dagger(k) a_s(k) \\
& - \frac{e}{m} \sum_{s=1}^2 \int \psi^\dagger(p) \psi(p) \frac{d^3k}{\sqrt{2k}} \tilde{\rho}(k) p \cdot e_s(k) [a_s(k) + a_s^\dagger(k)], \tag{1}
\end{aligned}$$

$\tilde{\rho}(k)$  is a smooth function that prevents ultraviolet divergences but does not affect infrared divergences, in particular  $\tilde{\rho}(0) = 1$ , and the fields are at time 0. For much of the discussion I will suppress the external potential  $V$ .

It is straightforward to see that the exact single-electron eigenstates are

$$|P\rangle = \sqrt{Z(p)} \exp\left\{ \frac{e}{m} \sum_{s=1}^2 \int d^3k \frac{\tilde{\rho}(k)}{\sqrt{2k}} p \cdot e_s(k) a_s^\dagger(k) \right\} \psi^\dagger(p) |0\rangle, \tag{2}$$

---

<sup>2</sup>This approximation is also made in [8]

$\psi(p)$  is the Fourier transform of  $\psi(x)$ , with energy

$$[\frac{1}{2m} - \frac{2e^2}{3m^2} \int_0^\infty dk \tilde{\rho}^2(k)] p^2 \equiv \frac{p^2}{2m(\infty)}. \quad (3)$$

(Fields and states in the physical space will be labelled with capital letters; vectors such as  $p$  will be without bold type or arrows.) The field strength renormalization factor is

$$\frac{1}{Z(p)} = \exp[\frac{e^2 p^2}{m^2} \int_0^\infty \frac{dk}{k} \tilde{\rho}^2(k)] \quad (4)$$

At first sight the transformation between  $|P\rangle$  and  $\psi^\dagger(p)|0\rangle$  does not seem to be even formally unitary. This first impression is misleading and is connected with the fact that annihilation operators disappear when acting on the vacuum. In fact the transformation between these states is

$$U = \exp\{\frac{e}{m} \int d^3p \psi^\dagger(p) \psi(p) d^3k \frac{\tilde{\rho}(k)}{\sqrt{2kk}} \sum_s p \cdot e_s(k) (a^\dagger(k) - a(k))\}. \quad (5)$$

Using the Baker-Hausdorff theorem, we find

$$\begin{aligned} U &= \exp\{-\frac{e^2 p^2}{2m^2} \int_0^\infty \frac{dk}{k} \tilde{\rho}^2(k)\} \exp\{\frac{e}{m} \int \frac{d^3k}{\sqrt{2kk}} \tilde{\rho}(k) p \cdot e^s(k) a_s^\dagger(k)\} \\ &\quad \exp\{-\frac{e}{m} \int \frac{d^3k}{\sqrt{2kk}} \tilde{\rho}^2(k) p \cdot e^s(k) a_s(k)\} \end{aligned} \quad (6)$$

Clearly when acting on the vacuum the factor with the annihilation operators becomes one and the remaining factors are not unitary. If one works in the algebra of operators rather than in the Hilbert space, one does not lose the annihilation parts of operators.

### 3 Calculation of the Heisenberg operator

The Heisenberg equations of motion are

$$-i\partial_t \psi^\dagger(p, t) = \frac{p^2}{2m} \psi^\dagger(p, t) - \frac{e}{m} \sum_{s=1}^2 \int \frac{d^3k}{\sqrt{2kk}} \tilde{\rho}(k) \psi^\dagger(p, t) p \cdot e_s(k) (a_s(k, t) + a_s^\dagger(k, t)) \quad (7)$$

$$-i\partial_t a_s^\dagger(k, t) = k a_s^\dagger(k, t) - \frac{e}{m} \int \frac{d^3 p}{\sqrt{2k}} \tilde{\rho}(k) \psi^\dagger(p, t) \psi(p, t) p \cdot e_s(k) \quad (8)$$

Since the Hamiltonian commutes with the product  $\psi^\dagger(p, t)\psi(p, t)$ , this product is conserved and the time can be dropped in the product when it appears in Eq.(8). Then the equation for  $a^\dagger$  can be solved exactly,

$$a_s^\dagger(k, t) = e^{ikt} a_s^\dagger(k) - i \frac{e}{m} \int \frac{d^3 p}{\sqrt{2k}} p \cdot e_s(k) \psi^\dagger(p) \psi(p) \tilde{\rho}(k) t. \quad (9)$$

The term in  $a^\dagger$  linear in  $t$  is connected with the neglect of momentum conservation in the model. This linear term cancels in the sum  $a^\dagger + a$  that appears in the equation of motion for  $\psi^\dagger$ . The solution for  $\Psi^\dagger$  is

$$\Psi^\dagger(p, t) = C \exp\left\{i \frac{p^2}{2m} t - \frac{e}{m} \sum_{s=1}^2 \int \frac{d^3 k}{\sqrt{2k}} \tilde{\rho}(k) p \cdot e_s(k) [e^{ikt} a_s^\dagger(k) - e^{-ikt} a_s(k)]\right\}. \quad (10)$$

Requiring  $\Psi^\dagger(p, 0) = \psi(p)$  gives

$$\Psi^\dagger(p, t) = \psi^\dagger(p) \exp\left\{i \frac{p^2}{2m} t - \frac{e}{m} \sum_{s=1}^2 \int \frac{d^3 k}{\sqrt{2k}} [f_s(p, k, t) a_s^\dagger(k) - f_s^*(p, k, t) a_s(k)]\right\}, \quad (11)$$

$$f_s(p, k, t) = \frac{\tilde{\rho}(p)}{\sqrt{2k}} p \cdot e_s(k) (e^{ikt} - 1). \quad (12)$$

The expectation of the number operator  $N = \sum_s \int d^3 k a_s^\dagger(k) a_s(k)$  for photons in the one electron state is

$$\langle N(t) \rangle = \frac{e^2}{m^2} \sum_s \int d^3 k |f_s(k, p, t)|^2 \quad (13)$$

$$= \frac{e^2}{m^2} \sum_s \int \frac{d^3 k}{2k^3} \tilde{\rho}^2(k) \sin^2 \frac{kt}{2}. \quad (14)$$

This is the same expression that Blanchard calls  $N^2(t)$ . As Blanchard shows, for all finite  $t$ ,  $\langle N(t) \rangle$  is finite, but for  $t \rightarrow \pm\infty$   $\langle N(t) \rangle$  diverges as  $\log t$ . Thus for all finite  $t$  there is a finite expectation value for the number of soft photons in the cloud

associated with the Heisenberg field of the electron, but for  $t \rightarrow \pm\infty$  the number diverges. A divergent value of  $\langle N \rangle$  is the signature for a space of vectors in an inequivalent representation of the commutation relations that is orthogonal to the original Fock space. The fact that the large- $t$  limit leads out of the usual Fock space has been identified as the cause of infrared divergences by several authors, starting with K.O. Friedrichs [10] and Blanchard [5].

The expectation value of the number operator also diverges in the exact eigenstates, Eq.(2). Thus the Heisenberg field acting on the vacuum creates states that are orthogonal to all the exact electron eigenstates in the model. This suggests that in this model one should introduce a field that makes the physical single particle state with its attached soft photons. In the single charged particle sector this should lead to diagonalization of the Hamiltonian.

## 4 Diagonalization of the Hamiltonian

We can invert the transformation  $U$  given in Eq.(5) to express the Hamiltonian in terms of “physical” operators that create the charged particles together with their soft photon clouds. We introduce the physical photon operators (and their adjoints),  $A(k) = Ua(k)U^\dagger$ ,

$$A(k) = a(k) - \frac{e}{m} \int d^3p \psi^\dagger(p) \psi(p) f_s(p, k) \quad (15)$$

as well as the transformed electron operators (and their adjoints),  $\Psi(p) = U\psi(p)U^\dagger$ ,

$$\Psi(p) = \exp\left\{\frac{e}{m} \int d^3k [f_s(p, k) a_s^\dagger(k) - f_s^*(p, k) a_s(k)]\right\} \psi(p). \quad (16)$$

In terms of the physical electron and photon field operators the Hamiltonian is

$$\begin{aligned}
H = & \Psi^\dagger(x) \left( -\frac{1}{2m(\infty)} \nabla^2 + V(x) \right) \Psi(x) + \sum_{s=1}^2 \int d^3k k A_s^\dagger(k) A_s(k) + \\
& + \sum_{s=1}^2 \int d^3p d^3p' f_s^*(p, k) f_s(p', k) \Psi^\dagger(p') \Psi^\dagger(p) \Psi(p') \Psi(p). \quad (17)
\end{aligned}$$

The bilinear terms in  $\Psi$  correspond to scattering of the electron together with its soft photon cloud in the external potential  $V$ . No infrared divergences or other infrared effects are visible. The model is taken to be a one electron model, so we ignore the quartic terms in  $\Psi(p)$ .

This result is similar to very old work on clothed operators in simple models of field theory [11], except in full QED we would propose to incorporate only the divergent part of the soft photons in the electron Heisenberg field, rather than eliminating all the trilinear terms in the original Hamiltonian.

## 5 Condition for the operator variational principle

In this model, the physical fields diagonalize the single-particle Hamiltonian. The asymptotic fields will be those that occur in scattering from the fixed potential  $V(x)$ . Thus the condition to be imposed on the Hamiltonian in the variational principle [9] based on choosing elements from the algebra of asymptotic fields should be to make the Hamiltonian as close to the free field Hamiltonian as possible. This condition is the analog of the condition chosen in theories without massless particles or fields.

## 6 What is relevant to QED?

Some issues suggested by this simple model may carry over to QED. The first is that one should express the observables of the theory in terms of a charged field that, acting on the vacuum, creates a state that is not orthogonal to the asymptotic states. The charged field should contain the soft photons in the neighborhood of



zero energy that are responsible for the infrared divergence in the number of perturbative photons attached to the charged particle. This can be done by a formally unitary transformation. The same unitary transformation should be applied to the photon field. Secondly, as suggested by Contopanagos and Einhorn [1], one can take the charged states with their clouds of soft coherent state photons to be in Fock space and the perturbative states to be in an inequivalent representation of the commutation relations.

In the present model, if one introduces the clothed electron operators in the theory the infrared divergences disappear and in the absence of an external potential the Hamiltonian has free field form because the physical photons decouple from the physical electrons. This of course will not happen in QED, nor will there be a discrete mass associated with the single charged particle. Since the work of B. Schroer [12] on “infraparticles” we have known that particles that interact with massless quanta cannot have discrete mass.

Since in the present model the single-particle Hamiltonian is diagonalized by the physical field, the variational principle is trivial in this model. When the massless quanta have been absorbed the variational principle proposed earlier for the case where there are no massless particles or fields can be applied. This also will not apply to QED and shows that significant modifications must be made in this variational principle before it can be useful in studying gauge theories.

## 7 Qualitative remarks

Divergent field strength renormalization occurs both due to ultraviolet effects and to infrared effects. In the case of ultraviolet divergences, one introduces the renormalized fields, for example  $\psi_{ren}(p) = Z^{-1/2}\psi(p)$ , where the multiplicative factor is fixed by requiring the coefficient of the  $\psi^{as}(p)$  term in the Haag expansion to be one; see references in [9]. This, together with other renormalization effects makes the Haag expansion finite. By contrast in the case of infrared divergences (suppressing non-infrared effects), one should not introduce a field strength renormalization; rather one should exponentiate the soft photons of near zero energy into a new operator-valued multiplicative factor multiplying the charged field as we have done

in Eq.(16). In the Pauli-Fierz model this explicitly removes the infrared divergences and produces a charged field that acting on the vacuum creates a state that is not orthogonal to all the asymptotic states. One can hope that in QED at least part of the infrared divergence can be removed using the physical charged field and that this field acting on the vacuum will produce asymptotic states.

One could argue<sup>3</sup> that given a Hamiltonian that has no trilinear term one could introduce trilinear terms in the transformed Hamiltonian and could then arrange to have any form of infrared divergence. My point of view is that, although highly simplified, this model does come from QED and does capture the type of soft photon infrared divergence that occurs in QED.

## 8 Outlook for future work

The idea of introducing physical charged fields that contain the infrared divergent part of the soft photons should be tried in more realistic models, such as the asymptotic Hamiltonian of Kulish and Faddeev [8], as well as in other models that have the same infrared divergences as QED.

### Acknowledgements

I am happy to thank Ted Jacobson, Xiang-Dong Ji and especially Ching-Hung Woo for helpful discussions.

## References

- [1] H.F. Contopanagos and M.B. Einhorn, Phys. Rev. D **45**, 1291 (1992).
- [2] R. Horan, M. Lavelle, and D. McMullan, Pramana **51**, 317 (1998).
- [3] E. d’Emilio and S. Micciché, hep-th/9908119.
- [4] W. Pauli and M. Fierz, Nuovo Cimento **15**, 167 (1938); reprinted in *Collected Scientific Papers by Wolfgang Pauli* Vol. 2, (Interscience, New York, 1964), ed. R. Kronig and V.F. Weisskopf, p813.

---

<sup>3</sup>Xiang-Dong Ji, private communication

- [5] P. Blanchard, Commun. math. Phys. **15**, 156 (1969).
- [6] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
- [7] D.R. Yennie, S.C. Frautschi and H. Suura, An. Phys. (N.Y.) **13**, 379 (1961).
- [8] P.P. Kulish and L.D. Faddeev, Teor. Mat. Fiz. **4**, 153 (1970) [Theor. Math. Phys. **4**, 745 (1971)].
- [9] O.W. Greenberg, Phys. Rev. D **58**, 065004, 1-7 (1998). Earlier literature on the N quantum approximation can be traced from this paper.
- [10] K.O. Friedrichs, Comm. Pure and Applied Math **5**, 349 (1952).
- [11] O.W. Greenberg and S.S. Schweber, Nuovo Cimento **8**, 378 (1958).
- [12] B. Schroer, Fortschr. Phys. **11**, 1 (1963).